FINAL-SEMESTER EXAMINATION
B. MATH III YEAR, II SEMESTER April 2016
ANALYSIS IV.

1. For $n \geq 1$, define s_n in $C_{\mathbb{R}}[0,1]$, the space of real continuous functions, as

$$s_n(x) = nx(1-x)^n.$$

Show that s_n converges to a function in $C_{\mathbb{R}}[0,1]$. Is the convergence uniform?

Solution: For $x = 0, 1$, the convergence is clear. For other x, use the ratio test x_{n+1}/x_n converges to a number $r < 1$, then x_n converges to 0. Hence $s_n(x)$ converges to the zero function pointwise, but not uniformly. For example take $x = 1/m$ for positive integer m and get a contradiction.

2. Let M be the subset of all functions f in $C_{\mathbb{R}}[0,1]$ with $f(0) = 0$ and $f(1) = 1$. Define $\hat{f}$ by

$$\begin{aligned} \hat{f}(x) :=& \frac{1}{2} f(3x) \quad for \quad 0 \leq x \leq 1/3 \\ & 1/2 \quad for \quad \frac{1}{3} \leq x \leq 2/3 \\ & \frac{1}{2}(1 + f(3x-2)) \quad for \quad 2/3 \leq x \leq 1. \end{aligned}$$

Show that there is a unique f in M such that $\hat{f} = f$.

Solution: Show that the map $f \to \hat{f}$ is contration and M is a complete metric space. The result will follow from Banach contraction principle.

3. Let D be the set of all $f \in C_{\mathbb{R}}[0,1]$ such that $|f(x) - x| \leq 1$ for all x. Prove or disprove the compactness of D.

Solution: In an infinite dimensional Banach space, the unit ball is non-compact. The essential argument here is the Arzela-Ascoli Theorem (see, Simmons).

4. Let $\mathcal{F}$ be an untra-filter on the set of natural numbers $\mathbb{N}$. Show that $\mathcal{F}$ contains a co-finite filter if and only if it doesn't contain a finite set.

Solution: See the previous midsemester question paper.

5. Consider the power series $p(x) = \sum_{n \geq 0} a_n (x - x_0)^n$ and take

$$\frac{1}{R} = \limsup_n |a_n|^{1/n}.$$

Show that the power series converges in $|x - x_0| < R$.

Solution: This is a standard theorem. See, any book on series, in particular Rudin's Principles of Mathematical Analysis.

6. Let f and g be two trigonometric polynomials, i.e.

$$f(x) = \sum_{N}^{N} a_n e^{2\pi nx}, \quad g(x) = \sum_{N}^{N} b_n e^{2\pi nx}.$$

Show that the Fourier coefficients satisfy the convolution property: $f \hat{*} g(n) = \hat{g}(n)\hat{g}(n)$.

Solution: See the proof given in Kreyzig Advanvced engineering Mathematics.

7. Compute the Fourier coefficients of the 1-periodic functions:

(1) $g(x) = |x - \frac{1}{2}|$ for $0 < x \leq 1$.

(2)

$$\begin{aligned} h(x) :=& 1 \quad for \quad 0 \leq x \leq 1/2 \\ & 0 \quad for \quad 1/2 \leq x < 1. \end{aligned}$$

Solution: Straightforward calculations. See the prevous midsem question papers. Also see, Kreyzig.